Quantitative Geometry of Loop Spaces

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April 24, 2019

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  - induces another notion of size on chains: supLength

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- **Question**: Given $\phi \in H_n(\Omega X)$, which parts of the volume/supLength plane do representatives live in?
- **Theorem**: S^3 triangulated with N 3-simplices, inducing cell structure on $\Omega_{PL}S^3$. Then any cellular sweepout $\Sigma \to \Omega_{PL}S^3$ requires $\gtrsim N^{4/3}$ 2-cells.

Thanks for listening!